The view point from the connection between orbital angular momentum and rotational frequency shift

Q. Wang*
School of Electrical & Electronic Engineering, Nanyang Technological University, Nanyang Avenue, Singapore 639798
**Corresponding author: wangqin@pmail.ntu.edu.sg*

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A unique thought experiment, which is based on the interference patterns between Laguerre-Gaussian beams and Gaussian beams, intuitively conveys the insight of the connection between orbital angular momentum and rotational frequency shift. From the view point of the connection, the existence of the orbital angular momentum can be ascertained as the natural result of the rotational frequency shift, and the formula of orbital angular momentum originally calculated through the integration of angular momentum density over the whole profile of Laguerre-Gaussian beam can be obtained through a simple algebra process. Moreover, from this point of view, the difficulties in the experimental verification of orbital angular momentum arising from the measurement of the feeble torque exerted on a mode convertor can be circumvented by measuring the rotational frequency shift instead. It is also pointed out that the similar way can be applied for the determination of the momentum of the photon for circularly polarized waves or plane waves.

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Under the paraxial approximation, electromagnetic waves in free space are governed by paraxial wave equation. In rectangular coordinates, the solutions to the paraxial equation are Hermite-Gaussian (HG) modes; while in cylindrical coordinates, the solutions are Laguerre-Gaussian (LG) modes [1].

In 1992, Allen *et al.* [2] presented the result that the photons of Laguerre-Gaussian mode possess a well defined orbital angular momentum (OAM) of $M = l\hbar$, where l is the azimuthal index, $\hbar$ is the reduced Planck constant. As shown in the article, the formula of OAM is obtained through the integration of the angular momentum density over the whole beam profile of the Laguerre-Gaussian mode, and it is indicated that the realization of the existence of the orbital angular momentum was inspired by the analogy between quantum mechanics and paraxial optics. Moreover, a method of measuring the mechanical torque of OAM by use of a suspended mode convertor is suggested in the article. Since its publication, this article has triggered considerable research interest in the area relating to orbital angular momentum.

Here we provide a view point from the connection between orbital angular momentum and rotational frequency shift (RFS), which can be drawn from a thought experiment based on the interference patterns between Laguerre-Gaussian beams and Gaussian beams—a manifestation of the rotational frequency shift resulted from the transfer of orbital angular momentum. From this view point, the existence of the orbital angular momentum is a natural result from the azimuthal angular dependent item in the mathematic expression of the Laguerre-Gaussian mode, and the formula $M = l\hbar$ can be readily obtained through a simple algebra process.

The suspended mode convertor approach for the measurement of mechanic torque arising from OAM suggested in [2] has to deal with considerable experimental difficulties in the measurement of the feeble mechanical torque. However, from above mentioned point of view, this difficulty can be circumvented by measuring the rotational frequency shift instead.

According to the mathematic expression of the LG solutions [1], there is an azimuthal angular dependent item of $\exp(jl\phi)$, where l is the azimuthal index, ϕ is the azimuthal angle. If there is a relative rotational movement with angular velocity of Ω between a Laguerre-Gaussian beam and an observer, the item $\exp(jl\phi)$ becomes $\exp(jl\Omega t)$, which means that the frequency of the Laguerre-Gaussian beam has shifted a value of $l\Omega$, and we can ascertain that the value of RFS is $l\Omega$.

Moreover, this rotational frequency shift can be experimentally measured by considering a simple thought experiment based on the interference patterns between Laguerre-Gaussian beams and Gaussian beams. Here the fan-like interference patterns as shown in Fig. 1 are taken as an example [3]. Imagine a detector with a small off-axis aperture rotating around the beam axis at an angular velocity of Ω, since the number of radial spokes of each interference pattern is equal to the azimuthal index l of the corresponding Laguerre-Gaussian mode, the signal frequency detected must be equal to $l\Omega$. And the frequency detected must be the frequency difference between the Laguerre-Gaussian mode and the coherent Gaussian mode. The Gaussian mode is symmetrical to the rotational movement around its axis, thus, its frequency seen by the rotational detector does not change, and we can ascertain that the frequency detected must be the RFS, i.e., RFS=$l\Omega$.

The existence of RFS means the energy of the photon of the optical vortex detected by the imaging rotational detector is changed; and the energy change of each photon is equal to $\hbar l\Omega$. According to the law of conservation of energy, there must be energy exchange between the light beam and the rotational detector. For the rotational detector side, assuming there are n photons being detected during a time interval Δt, and during this time interval, the detector rotates by an angle of $\Delta\theta$ ($\Delta\theta = \Omega\Delta t$). If M denotes the OAM of each photon, the angular momentum obtained by the detector is nM (The disregard of the direction will not affect the final conclusion), and the corresponding torque exerted on the detector is $nM / \Delta t$, which dose work of $nM\Omega$ on the rotating detector. According to the law of conservation of energy, the work done to the detector should be equal to the energy change of the photons. So we have $nM\Omega = n\hbar l\Omega$, i.e., $M = l\hbar$, which is exactly the result deduced in [2].

Comparing with other experimental approaches in the publications for the observation and measurement of RFS of Laguerre-Gaussian mode [4–6], the thought experiment shown here is simpler and more intuitive. As it can be seen, excepting the Laguerre-Gaussian beams and Gaussian beams employed for the interference, the thought experiment includes neither real rotating optical component nor optical detector, while they are indispensable in all the other experiments of its kind. And the conclusion about the value of RFS can be drawn straightforward from the interference patterns through the thought experiment. Moreover, the interference patterns between Laguerre-Gaussian mode and Gaussian mode could be considered as the first published experiment manifestation of RFS, as they have been known many years ago [7]—earlier than all the other publications shown the manifestation of RFS mentioned above.

Direct measurement of the torque arising from the spin angular momentum was completed by Beth [8] in 1966. The analogous experimental measurement for orbital angular momentum suggested in [2] has to deal with experimental difficulties in the measurement of the feeble torque exerted on the suspended mode convertor, and is not yet shown in any publications hitherto. Although the transfer of angular momentum from Laguerre-Gaussian beams to matter has been successfully observed by some research groups, as the quantitative ratio between the light energy absorbed and angular momentum gained by matter is almost impossible to be determined precisely in these experiments, the verification of the value of OAM is impossible either.

On the other hand, now that the connection between the OAM and RFS is dictated by the law of conservation of energy, the measurement of OAM is equivalent to the measurement of RFS. The later approach makes things much easier. In fact, it has already been done in the experiments shown in section 2 and elsewhere [4–6].

It should be mentioned that, the approach for the experimental measurement of orbital angular momentum suggested here is analogous to that used for the measurement of the angular momentum exerted by circularly polarized electromagnetic wave by P. J. Allen [9] in 1966. In which, the torque exerted by a circularly polarized electromagnetic wave was determined by measuring the frequency shift of the circularly polarized electromagnetic wave before and after the interaction with a rotating dipole, wherein, the connection between the angular momentum and frequency shift dictated by the law of conservation of energy, was exploited.

Besides, it is worth mentioning that, by applying the same view point, it can be easily verified that the similar relationship as that between the rotational frequency shift and the angular momentum of the photons for the Laguerre-Gaussian beams also holds for other kinds of optical beams, such as the circularly polarized light beams, as well as the plane waves. And in a similar way, it can be readily obtained both theoretically and experimentally that the spin angular momentum for the photon of circularly polarized light beam is $\hbar$, and the translational momentum for the photon of plane waves is h / λ, where λ is the wavelength.

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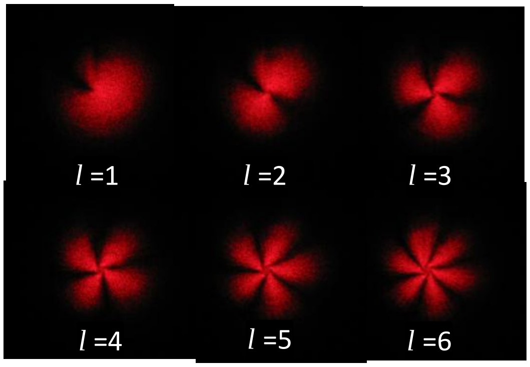

Fig.1. Interference patterns between Laguerre-Gaussian beams with azimuthal indices from 1 to 6 and Gaussian beams.

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