The view point from the connection between orbital angular momentum and rotational frequency shift

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A thought experiment based on the interference patterns between Laguerre-Gaussian beams and Gaussian beams shows the rotational frequency shift resulted from the transfer of the orbital angular momentum, from which the connection between orbital angular momentum and rotational frequency shift dictated by energy conservation law can be drawn. And from the view point of this connection, the existence of the orbital angular momentum of the Laguerre-Gaussian mode is the natural result of its azimuthal angle dependent item of $\exp(jl\phi)$, and the formula of orbital angular momentum $M = l\hbar$ previously calculated through the integration of angular momentum density over the whole profile of Laguerre-Gaussian beam [L. Allen *et al*, Phys. Rev. A **45**, 8185 (1992)] can be obtained through a simple algebra process. Moreover, from this point of view, the experiment difficulty in the measurement of the mechanical torque arising from orbital angular momentum can be circumvented by measuring the rotational

frequency shift instead. This view point provides a simple and intuitive way to understand the conceptions about the orbital angular momentum and rotational frequency shift, as well as the relationship between them.

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1. Introduction

Under the paraxial approximation, electromagnetic waves in free space are governed by paraxial wave equation. In rectangular coordinates, the solutions to the paraxial equation are Hermite-Gaussian (HG) modes; while in cylindrical coordinates, the solutions are Laguerre-Gaussian (LG) modes [1].

In 1992, Allen *et al* presented the result that the photons of Laguerre-Gaussian mode possess a well defined orbital angular momentum (OAM) of $M = l\hbar$ [2], where l is the azimuthal index, $\hbar$ is the reduced Planck constant. As shown in their article, the realization of the existence of the orbital angular momentum was inspired by the analogy between quantum mechanics and paraxial optics, and the formula is obtained through the integration of the angular momentum density over the whole beam profile of the Laguerre-Gaussian mode. Besides, a method of measuring the mechanical torque of OAM by use of a suspended mode convertor is suggested in their article. Since its publication, this article has triggered considerable research activities in the area relating to orbital angular momentum.

Here we provide a view point from the connection between orbital angular momentum and rotational frequency shift (RFS), which can be drawn from a thought experiment based on the interference patterns between Laguerre-Gaussian beams and Gaussian beams—a manifestation of the rotational frequency shift resulted from the transfer of orbital angular momentum. From this view point, the existence of the orbital angular momentum of the Laguerre-Gaussian mode is a natural result from its azimuthal dependent item of $\exp(jl\phi)$, and the formula $M = l\hbar$ can be readily obtained through a simple algebra process.

The suspended mode convertor approach for the measurement of mechanic torque arising from OAM suggested by Allen *et al* has to deal with considerable difficulty in the measurement of the feeble mechanical torque. However, from above mentioned point of view, this difficulty can be circumvented by measuring the rotational frequency shift instead.

2. The existence of the rotational frequency shift

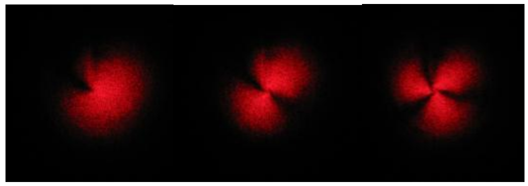
The mathematic expression of the LG solutions is stated as following [1]:

$$u_{pl}(\rho,\phi,z) = C_{pl}\cdot(\frac{1}{w})\exp(-i\frac{k\rho^2}{2R})\exp(-\frac{\rho^2}{w^2})\exp(-i(2p+l+1)\psi)\exp(-il\phi)\cdot(-1)^p(\frac{\sqrt{2}\rho}{w})^l L_p^l(\frac{2\rho^2}{w^2}) \qquad (2.2)$$

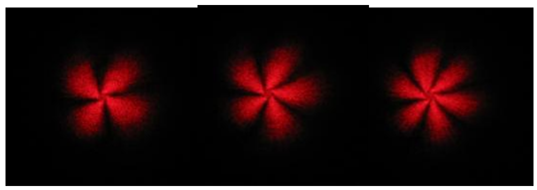
where u is the complex scalar wave amplitude which describes the transverse profile of the beam. p is called the radial index, l is called the azimuthal index, they can be either zero or any positive integer. z_R is the Rayleigh range, $R(z) = (z_R^2 + z^2)/z$, $\frac{1}{2}kw^2(z) = (z_R^2 + z^2)/z_R$, $\psi(z) = \arctan(z/z_R)$, C_{pl} is a normalization constant, $k = 2\pi/\lambda$ is the wave number, ρ,ϕ and z are parameters of cylindrical coordinates, $L_p^l(x)$ is the generalized Laguerre polynomial with $L_n^s(x) = \frac{e^x}{x^s} \cdot \frac{d^n}{dx^n}(e^{-x}x^{n+s})$ [3].

It can be seen that there is an azimuthal angular dependent item of $\exp(jl\phi)$ in the expression of Laguerre-Gaussian mode, where l is the azimuthal index, ϕ is the azimuthal angle. And when there is a rotational movement between the Laguerre-Gaussian mode and an observer, the factor $\exp(jl\phi)$ becomes $\exp(jl\Omega t)$, assuming Ω is the angular velocity of the rotation. This means the frequency of the Laguerre-Gaussian mode has shifted a value of $l\Omega$, and we can ascertain that the value of RFS is $l\Omega$.

This rotational frequency shift can be experimentally measured by considering a simple thought experiment based on the interference patterns between Laguerre-Gaussian beams and Gaussian beams. Here the fan-like interference patterns as shown in Fig. 1 are taken as an example [4]. Imagine a detector with a small off-axis aperture rotating around the beam axis at an angular velocity of Ω, since the number of radial spokes of each interference pattern is equal to the azimuthal index l of the corresponding Laguerre-Gaussian mode, the signal frequency detected must be equal to $l\Omega$. And the frequency detected must be the frequency difference between the Laguerre-Gaussian mode and the coherent Gaussian mode. The Gaussian mode is symmetrical to the rotational movement around its axis, thus, its frequency seen by the rotational detector does not change, and we can ascertain that the frequency detected must be the RFS, i.e., RFS= $l\Omega$.



$l = 1$ $l = 2$ $l = 3$



$l = 4$ $l = 5$ $l = 6$

Figure 1. Interference patterns between Laguerre-Gaussian beams with azimuthal indices from 1 to 6 and Gaussian beams.

2. The view point from the connection between OAM and RFS

The existence of RFS means the energy of the photon of the optical vortex detected by the imaging rotational detector is changed; and the energy change of each photon is equal to $h(\frac{l\Omega}{2\pi})$. According to the law of conservation of energy, there must be energy exchange between the light beam and the rotational detector. For the rotational detector side, assuming there are n photons being detected during a time interval Δt, and during this time interval, the detector rotates by an angle of $\Delta\theta$ ($\Delta\theta = \Omega\Delta t$). If M denotes the OAM of each photon, the angular momentum obtained by the detector is nM (The disregard of the direction will not affect the final conclusion), and the corresponding torque exerted on the detector is $\frac{nM}{\Delta t}$, which dose work of $nM\Omega$ on the rotating detector. According to the law of conservation of energy, the work done to the detector should be equal to the energy change of the photons. So we have $nh(\frac{l\Omega}{2\pi}) = nM\Omega$, i.e., $M = l\hbar$, which is exactly the result deduced by Allen *et al*.

3. Discussions

Comparing with other experimental approaches in the publications for the observation and measurement of RFS of Laguerre-Gaussian mode [5-7], the thought experiment shown here is simpler and more intuitive. As it can be seen, excepting the Laguerre-Gaussian beams and Gaussian beams employed for the interference, the thought experiment includes neither real rotating optical component nor optical detector, in the contrast, they are indispensable in all the other experiments of its kind. Moreover, the interference patterns between Laguerre-Gaussian mode and Gaussian mode could be considered as the first experiment manifestation of RFS, as they have been known many years ago [8]—much earlier than the other publications showing the manifestation of RFS.

Direct measurement of the torque arising from the spin angular momentum was completed by Beth in 1966 [9]. The analogous experimental measurement for orbital angular momentum which is suggested by Allen *et al* [2] is not yet shown in any publications hitherto partially due to the difficulty in the measurement of the feeble torque exerted on the suspended mode convertor. Although the transfer of angular momentum from Laguerre-Gaussian beam to matter has been successfully observed by some research groups, obviously, the quantitative ratio between the light energy absorbed and angular momentum gained by matter is almost impossible to be determined precisely in these experiments.

As shown in the view point, now that the connection between the OAM and RFS is dictated by the energy conservation law, the measurement of RFS is equivalent to the measurement of OAM. Apparently, this approach makes things much easier.

It should be mentioned that, the approach for the experimental measurement of orbital angular momentum suggested here is quite similar to that used for the measurement of the angular momentum exerted by circularly polarized electromagnetic wave by P. J. Allen in 1966 [10]. In which, the torque exerted by a circularly polarized electromagnetic wave was determined by measuring the frequency shift of the circularly polarized electromagnetic wave before and after the interaction with a rotating dipole, where the connection between the angular momentum and frequency shift was exploited.

Besides, it is worth mentioning that, by applying the same view point, it can be easily verified that the similar relationship as that between the rotational frequency shift and the angular momentum of the photons for the Laguerre-Gaussian beams also holds for other kinds of optical beams, such as the circularly polarized light beams, as well as the plane waves. In fact, from this view point, it can be readily obtained that the spin angular momentum for a photon of circularly polarized light beam is $\hbar$, and the translational momentum for the photon of plane waves is $\frac{h}{\lambda}$.

4. Conclusions

From the view point of the connection between orbital angular momentum and rotational frequency shift, the existence of orbital angular momentum is a natural result because of the azimuthal angular dependent item $\exp(jl\phi)$ of the Laguerre-Gaussian mode, and the formula of orbital angular momentum $M = l\hbar$ is just a result dictated by energy conservation law. And from this point of view, the experimental measurement of OAM is equivalent to the measurement of RFS, thus the experiment difficulty in the measurement of the feeble mechanical torque arising from OAM can be circumvented by measuring the RFS instead. Similar method can also be applied to the circularly polarized light beams and plane waves for the determination of spin angular momentum and translational momentum of photons, respectively.

Reference:

1. A. E. Siegman, Lasers, chapter 16, (University Science Books, c1986).
2. L. Allen, M. W. Beijersberjen, R. J. C. Spreeuw, and J. P. Woerdman, “Orbital angular momentum of light and the transformation of Laguerrre-Gaussian laser modes”, Phys. Rev. A 45, 8185−8189 (1992).
3. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 799-802, 1965.
4. Q. Wang, X. W. Sun, P. Shum and X. J. Yin, “Dynamic switching of optical vortices with dynamic gamma-correction liquid crystal spiral phase plate,” Opt. Express **13,** 10285−10291 (2005).
5. J. Courtial, K. Dholakia, D. A. Robertson, L. Allen, and M. J. Padgett, “Measurement of the rotational frequency shift imparted to a rotating light beam possessing orbital angular momentum,” Phys. Rev. Lett. 80, 3217−3219 (1998).
6. I. V. Basistiy, A. Ya. Bekshaev, M. V. Vasnetsov, V. V. Slyusar, and M. S. Soskin, “Observation of the Rotational Doppler Effect for Optical Beams with Helical Wave Front Using Spiral Zone Plate,” JETP Lett. **76**, 486−488 (2002).
7. I. V. Basistiy, V. V. Slyusar, M. S. Soskin, and M. V. Vasnetsov, “Manifestation of the rotational Doppler effect by use of an off-axis optical vortex beam,” Opt. Lett. **28,** 1185 −1187 (2003).
8. A. G. White and C. P. Smith *et al*, “Interferometric measurements of phase singularities in the output of a visible laser,” Journ. of Mod. Opt., 38, 2531−2541(1991).
9. Rechard Beth, “Mechanical detection and measurement of the angular momentum of light,” Phys. Rev. **50**, 115−125 (1936).

10. P. J. Allen, “A Radiation Torque Experiment ,” Am. J. Phys. **34**, 1185-1192 (1966).